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ON THE STABILITY OF WEAKLY INHOMOGENEOUS STATES WITH A SMALL ADDITION OF WHITE NOISE

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Using the concepts developed in [1] we investigate, in the presence of certain restrictions, the stability of a weakly inhomogeneous state parametrically perturbed by a small random addition of white noise. We show that when the characteristic wavelength is arbitrarily small as compared with the distance over which it varies substantially, then the mechanism of formation of the eigenfunctions responsible for the stability of the state is analogous to the mechanism given in [1]. In the present case it is not the boundaries that act as reflectors, as in [1], but the points at which the condition of existence of the global eigenfunction for the homogeneous problem holds. We obtain the criterion of stability of the state in question and discuss the problem of application of the results obtained to the case in which the ratio of the characteristic wavelength to the distance over which it varies substantially, cannot be taken as arbitrarily small.

1. The statement of the problem is analogous to that given in [1]. We consider the following homogeneous problem:

$$\sum_{ikm}\left[a_{ikml}(\varepsilon x)+hd_{ikml}(\varepsilon x)\,F\left(\frac{x}{\delta}\right)\right]D_{ik}\Psi_m=0,\qquad D_{ik}=\frac{\partial^i}{(\partial x)^i}\,\frac{\partial^k}{(\partial t)^k}\quad(1.1)$$

$$\Big(\sum_{ikm} f_{ikml}D_{ik}\Psi_m\Big)_{x=0},\qquad l=1,2,\ldots,n$$

$$\Big(\sum_{ikm} f_{ikml}D_{ik}\Psi_m\Big)_{x=L/\varepsilon},\qquad l=n+1,\ldots,N,\ L\sim 1$$

Here N is the order of the system (1.1) with respect to x, a_{ikml} and d_{ikml} are functions with the characteristic scales and moduli of order of unity. F is a real stationary random function with characteristic scale and modulus of the order of unity. ε, h and δ are small real parameters and

$$\langle F(\theta)\rangle = 0, \quad \langle F(\theta) F(\theta')\rangle = \exp[-(\theta - \theta')^2] \tag{1.2}$$
$$\varepsilon^{\alpha} < h < \varepsilon, \quad \varepsilon^{\alpha} < \delta < \varepsilon^{\alpha_1}, \quad \alpha > \alpha_1 > 0$$

where the angular brackets denote averaging over the ensemble.

We assume that the dispersion relation for the system (1.1) without the right-hand side term contained within the square bracket

$$\det\Big[\sum_{ik} a_{ikml}(\varepsilon x)\,\lambda^i p^k\Big] = 0 \tag{1.3}$$

satisfies, on the real axis, the Petrovskii condition uniformly in x [1, 2]. This means that p_0 exists such, that $\operatorname{Re}\lambda_i < 0$ $(i = 1, \ldots, s)$ and $\operatorname{Re}\lambda_i > 0$ $(i = s+1, \ldots, N)$ when $\operatorname{Re} p > \operatorname{Re} p_0$ uniformly in x. The problem is assumed to be stated correctly [1, 3], i.e. there are s boundary conditions at the zero and $N - s$ boundary conditions at $x = L/\varepsilon$. It is also assumed that the form of the operator in (1.1) is such, that the problem of solution stability can be reduced to that of finding the solutions of the form $\Psi_m = \exp(pt)\ \Psi_m(x)$.

2. After the substitution $\Psi_m = \exp(pt)\,\Psi_m(x)$ and change of the unknowns the problem (1.1) reduces to the following:

$$y_i' - \frac{\lambda_i}{\varepsilon} y_i = c_{ik} y_k + \frac{h}{\varepsilon} d_{ik} F\left(\frac{x}{\varepsilon\delta}\right) y_k \tag{2.1}$$

$$b_{ki} y_i(0) = 0, \quad k = 1, \ldots, s \tag{2.2}$$
$$b_{ki} y_i(L) = 0, \quad k = s+1, \ldots, N$$

in which x have been replaced by x/ε and $\lambda_i(x, p)$ are roots of Eq. (1.3).

Let us consider a family of curves $\operatorname{Re}(\lambda_i - \lambda_k) = 0$ $(i \leqslant s,\ k > s)$ in the p-plane. When x varies continuously over the interval (x_1, x_2), the curves fill a certain region to the left of p_0. Let us denote by $\Gamma(x_1, x_2)$ the envelope of this family. We assume that $\Gamma(0, L)$ or at least the part of $\Gamma(0, L)$ lying near the largest $p_r = \operatorname{Re} p$, has the following property: $\lambda_i(p) \neq \lambda_k(p)$ $(i \leqslant s,\ k > s,\ p \in \Gamma(0, L))$.

The following assertion will be needed in the subsequent analysis.

Assertion 1. Let $\Delta > 0$ exist such that

$$\max\nolimits_{i\leqslant r} \operatorname{Re}\lambda_i + 3\Delta \leqslant \min\nolimits_{i>r} \operatorname{Re}\lambda_i$$

uniformly in x, $x \in [x_1, x_2]$. Then for the solution of (2.1) such that

$$y_{im}(x_1) = \delta_{im} \quad (i \leqslant r,\ m \leqslant r), \quad y_{im}(x_2) = 0 \quad (i > r,\ m \leqslant r)$$

the estimates

$$|y_{im}(x_2)| \leqslant 2I_1(x_1, x_2), \quad |y_{im}(x_1)| \leqslant 2\varepsilon M/\Delta \tag{2.3}$$

$$I_1(x_1, x_2) = \exp\left(\frac{1}{\varepsilon}\int_{x_1}^{x_2} f_1\,dx\right), \quad f_1 = \max\nolimits_{i<r} \operatorname{Re}\lambda_i + \Delta$$

$$M = \max\Big(\max\nolimits_i \sum_k \max\nolimits_x |c_{ik}|,\ \max\nolimits_i \sum_k \max\nolimits_x |d_{ik}|\Big)$$

hold, and they are proved as follows. Integration and substitution $y_i = u_i I_1(x_1, x)$ reduces (2.1) to the form

$$u_i = u_i^{(0)} + U_{ik}(x_1, x)\, u_k, \quad i \leqslant r, \qquad u_i = -U_{ik}(x_2, x)\, u_k, \quad i > r$$

$$U_{ik}(x_1, x) = \int_{x_1}^{x} \left(c_{ik} + \frac{h}{\varepsilon} d_{ik} F\right) \exp\left[\frac{1}{\varepsilon}\int_t^x (\lambda_i - f_1)\, dt\right] dt$$

$$u_i^{(0)} = \delta_{im} \exp\left(\frac{1}{\varepsilon}\int_{x_1}^{x} (\lambda_i - f_1)\, dx\right)$$

The functions u_i are sought in the form $u_i = u_i^{(0)} + u_i^{(1)} + \ldots$. The estimates for $u_i^{(n)}$ are easily obtained and the estimates given above for y_{im} follow from them. In the same manner we show that for the solutions of (2.1) such that

$$y_{im}(x_2) = \delta_{im}, \quad i > r, \quad m > r; \qquad y_{im}(x_1) = 0, \quad i \leqslant r, m > r$$

the estimates

$$|y_{im}(x_1)| \leqslant 2I_2(x_2, x_1), \qquad |y_{im}(x_2)| \leqslant 2\varepsilon M / \Delta \tag{2.4}$$

$$I_2(x_2, x_1) = \exp\left(\frac{1}{\varepsilon}\int_{x_2}^{x_1} f_2 dx\right), \qquad f_2 = \min_{i>r} \operatorname{Re}\lambda_i - \Delta$$

hold. Using the estimates (2.3) and (2.4) we can prove the next assertion which will be necessary in what follows.

Assertion 2. Let p lie to the right of $\Gamma(0, a)$ and $\det(b_{ik}(p)) \neq 0$ $(i \leqslant s, k \leqslant s)$. Then the boundary conditions (2.2) can be transferred from 0 to a, where they will assume the form

$$(\delta_{ik} + \varepsilon T_{ik})\, y_k(a) = 0, \quad i \leqslant s \quad (|T_{ik}| \lesssim 1)$$

The boundary conditions at L / ε can be transferred to the left under the similar conditions. When p lies to the right of $\Gamma(0, L)$, we have

$$\max_{i \leqslant s} \operatorname{Re}\lambda_i + 3\Delta \leqslant \min_{i > s} \operatorname{Re}\lambda_i$$

Writing the equations for the eigenvalues in the determinant form [1] and using the estimates of Assertion 1, we can obtain the next assertion.

Assertion 3. A value of p lying to the right of $\Gamma(0, L)$ can be an eigenvalue of the problem (2.1), (2.2) if and only if

$$\det(b_{ik}(p)) = 0, \quad i \leqslant s, \quad k \leqslant s \quad \text{or} \quad \det(b_{ik}(p)) = 0, \quad i > s, \quad k > s$$

The instability generated by such eigenvalues is called limiting instability [1].

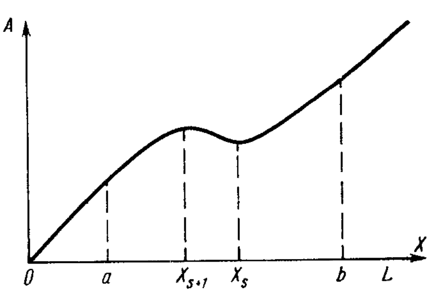

Fig. 1

Fig. 1 depicts the behavior of the quantity

$$A = \operatorname{Re}\int_0^x (\lambda_{s+1} - \lambda_s)\, dt$$

for the values of p lying to the left of $\Gamma(0, L)$, sufficiently close to the points on the curve $\operatorname{Re}(\lambda_s - \lambda_{s+1}) = 0$ belonging to $\Gamma(0, L)$. On a certain interval $[a, b]$ such that

$$\operatorname{Re}\int_a^{x_s}(\lambda_{s+1}-\lambda_s)\,dt>0,\quad \operatorname{Re}\int_{x_{s+1}}^{b}(\lambda_{s+1}-\lambda_s)>0$$

the following conditions hold:

$$\max\nolimits_{i<s}\operatorname{Re}\lambda_i+2\Delta\leqslant\operatorname{Re}\lambda_s\leqslant\min\nolimits_{i>s+1}\operatorname{Re}\lambda_i-2\Delta \tag{2.5}$$
$$\max\nolimits_{i<s}\operatorname{Re}\lambda_i+2\Delta\leqslant\operatorname{Re}\lambda_{s+1}\leqslant\min\nolimits_{i>s+1}\operatorname{Re}\lambda_i-2\Delta$$

For such values of p the estimates (2.3) and (2.4) become insufficient for the study of the equations for the eigenvalues. Two solutions of the system (2.1), namely s and $s+1$, are found in the form of series in h/ε. We assume that when $h=0$, the system (2.1) has solutions in $[a, b]$ of the form

$$\exp\left(\frac{1}{\varepsilon}\int_a^x\lambda_s dt\right)(\delta_{is}+\varepsilon B_{is}),\quad \exp\left(\frac{1}{\varepsilon}\int_b^x\lambda_{s+1}dt\right)(\delta_{is+1}+\varepsilon B_{is+1})$$

If p lies sufficiently close to $\Gamma\ (0, L)$, then solutions of this form obviously exist

$$y_{is}(a)=\delta_{is},\quad i\leqslant s;\quad y_{is}(b)=0,\quad i>s+1;\quad y_{s+1s}(x_{s+1})=0 \tag{2.6}$$
$$y_{is}(x)=\exp\left(\frac{1}{\varepsilon}\int_a^x\lambda_s dt\right)(\delta_{is}+\varepsilon B_{is})+$$
$$\exp\left(\frac{1}{\varepsilon}\int_a^x\lambda_{0s}dt\right)\sum_{l=1}^{\infty}\left(\frac{h}{\varepsilon}\right)^l\int_a^b\cdots\int_a^b F(t_1)\ldots F(t_l)\,\Phi_{ils}dt_l\ldots dt_1$$

$$\lambda_{0s}=\begin{cases}\lambda_s, & x\leqslant x_s\\ \lambda_{s+1}, & x>x_s,\end{cases}\qquad |\Phi_{ils}|\leqslant 2(M+2M^2)\exp[lM(b-a)]$$

$$\int_a^b F\Phi_{s+1\,1s}\,dt=\int_{x_{s+1}}^{x}F\varphi(x_s, t)\,d_{s+1s}dt \tag{2.7}$$
$$\left|\int_a^b F\Phi_{i1s}dt\right|\leqslant\exp\left[\frac{1}{\varepsilon}\int_a^x\operatorname{Re}(\lambda_s-\lambda_{0s})\,dt\right],\quad i\neq s+1$$
$$\varphi(x_s, t)=\exp\left(-\frac{1}{\varepsilon}\int_{x_s}^{t}(\lambda_{s+1}-\lambda_s)\,dt\right)$$

The solution y_{is+1} can be obtained in the analogous manner, but s in the formulas (2.6) and (2.7) must be replaced by $s+1$, $s+1$ by s, a by b and b by a. In addition, the signs of all inequalities in (2.6) except that for Φ_{ils}, must be reversed.

When $\varepsilon\to 0$, the random quantities $y_{s+1s}(b)$ and $y_{ss+1}(a)$ tend to their root mean square values [4]

$$z_{s+1s}=\exp\left(\frac{1}{\varepsilon}\int_a^b\lambda_{0s}dt\right)z_1,\quad z_1=\frac{h}{\varepsilon}\int_{x_{s+1}}^{b}Fd_{s+1s}\varphi(x_s, t)\,dt \tag{2.8}$$

$$z_{ss+1}=\exp\left(\frac{1}{\varepsilon}\int_b^a\lambda_{0s+1}dt\right)z_2,\quad z_2=\frac{h}{\varepsilon}\int_{x_s}^{a}Fd_{ss+1}\varphi(t, x_{s+1})\,dt \tag{2.9}$$

i.e.

$$\frac{\langle | y_{s+1s}(b) - z_{s+1s}|^2\rangle}{\langle | z_{s+1s}|^2\rangle} \to 0, \qquad \frac{\langle | y_{ss+1}(a) - z_{ss+1}|^2\rangle}{\langle | z_{ss+1}|^2\rangle} \to 0 \tag{2.10}$$

When $\varepsilon \to 0$, (2.6), (2.7) and the form of the correlation function together readily yield

$$\langle | z_{s+1s}|^2\rangle \simeq \pi h^2 \delta \varepsilon^{-1/2} | d_{ss+1}(x_{s+1})|^2 \left(\frac{\partial \operatorname{Re}(\lambda_{s+1} - \lambda_s)}{\partial x}\right)^{-1/2}_{x=x_{s+1}} \exp\left(\frac{2}{\varepsilon}\operatorname{Re}\int_a^b \lambda_{0s}\, dt\right)$$

$$\langle | y_{s+1s} - z_{s+1s}|^2\rangle \sim h^3 \delta^2 \varepsilon^{-1/2} \exp\left(\frac{2}{\varepsilon}\operatorname{Re}\int_a^b \lambda_{0s}\, dt\right)$$

The above estimates yield the first formula of (2.10), and an entirely analogous result can also be obtained for the second formula.

3. When p lie to the left of $\Gamma(0, L)$ and sufficiently close to it, the boundary conditions can, according to Assertion 2, be transferred from 0 to a and from L to b and then the problem can be considered in the interval $[a, b]$. We can write the equation for the eigenvalues only if we know the complete system of solutions of (2.1) on a, b. Since the inequalities (2.5) hold on a, b, then $N-2$ solutions of (2.1) can be taken from Assertion 1, and for these solutions the estimates (2.3) and (2.4) will hold. In this case f_1 amd f_2 have the form

$$f_1 = \max_{i<s} \operatorname{Re}\lambda_i + \Delta, \qquad f_2 = \min_{i>s+1} \operatorname{Re}\lambda_i - \Delta$$

The remaining two solutions of (2.1) are given by the formulas (2.6) and (2.7). The equation defining the eigenvalues has the form

$$\begin{vmatrix} \begin{matrix} Q_{im}(a) \\ i \leqslant s,\ m \leqslant s \end{matrix} & & \begin{matrix} Q_{is+1}(a) \\ i \leqslant s \end{matrix}, & \begin{matrix} Q_{im}(a) \\ i \leqslant s,\ m > s+1 \end{matrix} \\ \begin{matrix} Q_{im}(b) \\ i > s,\ m < s \end{matrix}, & \begin{matrix} Q_{is}(b) \\ i > s \end{matrix} & & \begin{matrix} Q_{im}(b) \\ i > s,\ m > s \end{matrix} \end{vmatrix} = 0 \tag{3.1}$$

$$(Q_{il}(c) = (\delta_{ik} + \varepsilon T_{ik})\, y_{kl}(c))$$

and can be written as

$$1 - \varphi(x_{s+1}, x_s)\, z_1 z_2 + \gamma = 0 \tag{3.2}$$

From the estimates (2.3) and (2.4), the form of the s-th and ($s+1$)-th solutions and the formula (2.10) it follows that γ is small compared with the other two terms of (3.2). For this reason below we shall consider the equation

$$1 - \varphi(x_{s+1}, x_s)\, z_1, z_2 = 0 \tag{3.3}$$

which is equivalent to the system

$$\frac{|\varphi(x_s, x_{s-1})|^2}{\langle | z_1|^2\rangle \langle | z_2|^2\rangle} = \theta, \qquad \theta = \frac{|z_1|^2\, |z_2|^2}{\langle | z_1|^2\rangle \langle | z_2|^2\rangle} \tag{3.4}$$

$$\int_{x_{s+1}}^{x_s} \operatorname{Im}(\lambda_s - \lambda_{s+1})\, dt = \varepsilon k \pi \tag{3.5}$$

In (3.5) the small terms $\varepsilon\varphi_1$ and $\varepsilon\varphi_2$ are neglected; φ_1 and φ_2 are the arguments of

z_1 and z_2.

In what follows, we must learn the probability density of the quantity θ. When $\varepsilon \to 0$, the real and imaginary parts of z_1 and z_2 have the form of stochastic integrals and are distributed in accordance with the normal rule [4]. Any linear combination of the imaginary and real parts of z_1 as well as of z_2 has the form of stochastic integrals and is distributed normally. This implies that the two-point densities $\rho\,(\mathrm{Re} z_1,\ \mathrm{Im}\, z)$ have the normal form. Rearranging $\rho\,(\mathrm{Re} z_1,\ \mathrm{Im}\, z)$ we can obtain $\rho\,(|z|^2)$. Rearranging $\rho\,(|z_1|^2)$ and $\rho\,(|z_2|^2)$ and assuming that z_1 and z_2 are independent, we can easily obtain the required density $\rho\,(\theta)$ (z_1 is determined by the neighborhood of x_{s+1}, and z_2 by the neighborhood of x_s and F can be correlated at the distances of the order of $\varepsilon\delta$, therefore z_1 and z_2 can be assumed independent provided that $x_s - x_{s+1} \gg \varepsilon\delta$). Performing the computations based on the above arguments, we obtain

$$\rho\,(\theta) = \frac{1}{4}\int\limits_0^\infty \frac{1}{t}\exp\left(\frac{\theta}{4t} - t\right) dt \tag{3.6}$$

Expressing p_i from (3.5) in terms of p_r and substituting this into (3.4), we obtain the equation for p_r.

We introduce $\psi\,(p_r)$ in the following manner:

$$\psi\,(p_r) = \frac{|\,\varphi\,(x_s,\ x_{s+1})\,|^2}{\langle|\,z_1\,|^2\rangle\ \langle|\,z_2\,|^2\rangle} - \theta \tag{3.7}$$

We shall say that $\psi\,(p_r)$ has a zero in the interval $[p_r - \Delta p_r,\ p_r + \Delta p_r]$ if the probability that $\psi\,(p_r + \Delta p_r) > 0$ and $\psi\,(p_r - \Delta p_r) < 0$ is ($1 - \varepsilon$). Let us write the expression for $\psi\,(p_r)$ in the neighborhood of the point p_r in which the first term of (3.7) is unity

$$\psi\,(p_r + \Delta p_r) \simeq \exp\left(-\frac{2\Delta p_r}{\varepsilon} S\right) - \theta \tag{3.8}$$

$$S = \int\limits_{x_{s+1}}^{x_s} \frac{\partial\, \mathrm{Re}\,(\lambda_s - \lambda_{s+1})}{\partial p_r}\, dt$$

Let P_1 be the probability that $\psi\,(p_r + \Delta p_r^{(1)}) < 0$ and P_2 the probability that $\psi\,(p_r + \Delta p_r^{(2)}) > 0$

$$P_1 = \int\limits_{c_1}^{\infty} \rho\,(\theta)\, d\theta, \quad P_2 = \int\limits_0^{c_2} \rho\,(\theta)\, d\theta, \quad c_l = \exp\left(-\frac{2\Delta p_r^{(l)}}{\varepsilon} S\right), \quad l = 1, 2$$

Equations $P_1 = \varepsilon$ and $P_2 = \varepsilon$ yield $\Delta p_r^{(1)}$ and $\Delta p_r^{(2)}$

$$\Delta p_r^{(1)} \simeq -\,\varepsilon \ln|\ln \varepsilon|\, S^{-1}, \qquad \Delta p_r^{(2)} = -\,\varepsilon \ln \varepsilon S^{-1} \tag{3.9}$$

From (3.4), (3.5) and (3.9) it follows that the eigenvalues of the problem (2.1), (2.2) with the largest p_r, lie at the distance

$$|\,\Delta p_r\,| < |\varepsilon \ln \varepsilon\, S^{-1}| \tag{3.10}$$

from any point of the curve defined by the equation

$$|\varphi\,(x_s,\ x_{s+1})\,|^2\ \langle|\,z_1\,|^2\rangle\ \langle|\,z_2\,|^2\rangle = 1 \tag{3.11}$$

It can be shown, using (3.6), that the probability of the zeros (3.8) lying to the left and right of the curve (3.11), is of the order of unity. The relation (3.10) with $\varepsilon \to 0$

yields the strip with Δp_r and the distance between the curve of eigenvalues given by (3.11) and $\Gamma\ (0,\ L)\ \Delta p_{r0}$

$$\Delta p_r \sim \varepsilon^{2/3} \ln^{-1/3} (h^2 \delta \varepsilon^{-1/2}), \qquad \Delta p_{r0} = \varepsilon^{2/3} \ln^{2/3} (h \delta \varepsilon^{1/2}) \tag{3.12}$$

Thus when $\varepsilon \to 0$, the solution of (1.1) is stable if $\Gamma\ (0,\ L)$ lies in the left half-plane of p, and unstable if $\Gamma\ (0,\ L)$ enters the right half-plane.

With the exception of (3.12), the formulas derived above can be used in certain cases when ε is small but finite. This is possible when the condition $\mathrm{Re}\ (\lambda_i - \lambda_k) < 0\ (i \leqslant s,\ k > s)$ holds in the half-plane, $p_r > 0$ only for a single pair of values $(i = s,\ k = s + 1)$ (which ensures that the inequality (2.5) holds in some $[a,\ b]$ $(a < x_{s+1},\ b > x_s)$ and the system (2.1) with $h = 0$ has solutions of the form

$$\exp \left(\frac{1}{\varepsilon} \int\limits_a^x \lambda_s dx\right) (\delta_{is} + \varepsilon B_{is}), \quad \exp \left(\frac{1}{\varepsilon} \int\limits_b^x \lambda_{s+1} dx\right) (\delta_{is+1} + \varepsilon B_{is+1})$$

in $[a,\ b]$. This makes it possible to write the s and $(s + 1)$ solutions in the form (2.6). In this case the stability is determined by the position of the curve (3.11).

Example. We consider the stability of a rod in the form of a long, thin plate of variable width (length L / ε, width $l\ (\varepsilon x)$ and thickness $d \ll l$) moving through a gas at a very high supersonic speed (Fig. 2). The force acting on the unit surface of the rod is approximately [5]

$$q = B \left(\frac{\partial W}{\partial x} V - \frac{\partial W}{\partial t} \right)$$

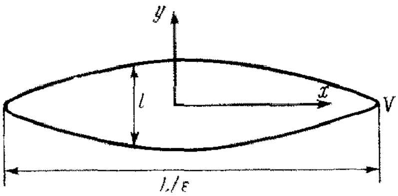

Fig. 2

where ($W\ (x,\ y,\ t)$ is the coordinate of the rod surface. When B is small (i.e. at low gas pressure), the effect of the force q on the transvese oscillations of the rod can be neglected and only its action on the torsional oscillations taken into account. This is related to the fact that the torsional moment M produced by the force q is proportional to l^3 and the torsional stiffness $c \sim l$. The simplification achieved by the above procedure does not alter the crux of the matter, but leads to a simpler dispersion relationship.

The oscillation equations have the form

$$\frac{\partial^2}{\partial x^2} \left(EJ\ \frac{\partial^2 y}{\partial x^2} \right) + m\ \frac{\partial^2 y}{\partial t^2} = -\ hF \left(\frac{x}{\delta} \right) m \frac{\partial^2 \varphi}{\partial t^2}$$

$$\frac{\partial}{\partial x} \left(c\ \frac{\partial \varphi}{\partial x} \right) - I\ \frac{\partial^2 \varphi}{\partial t^2} + \frac{2}{3}\ Bl^3 \left(\frac{\partial \varphi}{\partial x} V - \frac{\partial \varphi}{\partial t} \right) = hF \left(\frac{x}{\delta} \right) m \frac{\partial^2 y}{\partial t^2}$$

Here EJ is the flexural rigidity, c is torsional stiffness, I is the moment of the unit rod length relative to its center of inertia, m which is the mass of the unit rod length and l are both functions of εx, and $hF\ (x / \delta)$ is the distance between the center of gravity and the center of stiffness of the rod cross section (white noise). In general, the slowly varying functions may contain additional small random terms, but compared with $hF\ (x / \delta)$ they do not contribute anything new and are therefore neglected.

It can easily be shown that the system given above is formally identical to (1.1) and satisfies all necessary restrictions. Equation (1.3) for this case assumes the form

$$EJ\lambda^4 + mp^2 = 0, \quad c\lambda^2 + \frac{2}{3} Bl^3 (\lambda V - p) - Ip^2 = 0$$

$\Gamma(0, L)$ is found from equation

$$\mathrm{Re}\left[\frac{Bl^3}{3c}+\left(\frac{Bl^6V^2}{9c^2}+\frac{2}{3}\,\frac{Bl^3}{c}\,p+\frac{I}{c}\,p^2\right)^{1/2}+\left(-\frac{m}{EJ}\right)^{1/4}p^{1/2}\right]=0$$

and lies in the right half-plane if $\max_x (IV^2/c) > 1$. Thus a rod of sufficient length is stable if $\max_x (IV^2/c) < 1$ and unstable if $\max_x (IV^2/c) > 1$.

The concepts developed above can also be used in the problems which can be reduced to infinite systems of ordinary differential equations, such as e.g. the problems of hydrodynamic stability.

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ON THE "EQUIVALENCE" RULE FOR FLOWS OF PERFECT GAS

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An extension of the equivalence of "area" rule [1, 2] is presented. The rule was initially derived for stationary flows of perfect (inviscid and non-heat-conducting) gas past slender fine pointed bodies (or blunted bodies in the hypersonic flow case) whose transverse dimensions are small in comparison with their length. According to that rule the wave drag of a three-dimensional body is equal to the wave drag of an axisymmetric body with the same distribution of cross-sectional areas along the axis. The rule is extended here to stationary and nonstationary flows past nonslender bodies and to internal flows, using the procedure of averaging with respect to the angular variable of a cylindrical system of coordinates. That procedure is, strictly speaking, valid for nearly axisymmetric bodies. However the numerical solutions obtained by the authors for a fairly wide range of external and internal problems show that the generalized equivalence rule is applicable to substantially nonaxisymmetric configurations (*) (see next page).